

**GENERALIZED  $q$ -GAUSSIAN VON NEUMANN ALGEBRAS WITH  
COEFFICIENTS, III. UNIQUE PRIME FACTORIZATION RESULTS.**

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ABSTRACT. We prove some unique prime factorization results for tensor products of type  $II_1$  factors of the form  $\Gamma_q(\mathbb{C}, S \otimes H)$  arising from symmetric independent copies with sub-exponential dimensions of the spaces  $D_k(S)$  and  $\dim(H)$  finite and greater than a constant depending on  $q$ .

## 1. INTRODUCTION.

This article is a continuation of the program initiated in [2]. In [2] we introduced the generalized  $q$ -gaussian von Neumann algebras  $\Gamma_q(B, S \otimes H)$  with coefficients in  $B$  and proved their strong solidity relative to  $B$  under the assumptions of  $\dim_B(D_k(S))$  sub-exponential and  $\dim(H) < \infty$  (see [2], Def. 3.18 and Cor. 7.4). In a subsequent paper ([3]) we investigated the presence of non-trivial central sequences and we showed they do not exist when  $B$  is a finite dimensional factor, the dimensions over  $B$  of the modules  $D_k(S)$  are sub-exponential and the dimension of  $H$  is finite and greater than a constant depending on  $q$ . In the present work we prove some unique prime factorization results for tensor products of von Neumann algebras of the form  $\Gamma_q(\mathbb{C}, S \otimes H)$  arising from a sequence of symmetric independent copies over  $\mathbb{C}$  and having sub-exponential dimensions (over  $\mathbb{C}$ ) of the spaces  $D_k(S)$  introduced in [2], Def. 3.18. The first results of this kind for type  $II_1$  factors - arising from either (discrete) ICC non-amenable hyperbolic groups or (discrete) subgroups of connected simple Lie groups of rank one - have been obtained by Ozawa and Popa in [9], through a combination of Ozawa's  $C^*$ -algebraic techniques previously used in [8] and the powerful intertwining and unitary conjugacy techniques of Popa (see e.g. [10], Appendix and [11], Thm. 2.1). Let's recall that for  $(\mathcal{M}, \tau)$  a type  $II_1$  factor and  $t > 0$ , the amplification of  $\mathcal{M}$  by  $t$  is defined as  $\mathcal{M}^t = p(M_n(\mathbb{C}) \otimes \mathcal{M})p$ , where  $n > t$  and  $p \in M_n(\mathbb{C}) \otimes \mathcal{M}$  is a projection with  $\tau(p) = t/n$ . Our main result is (see also Thm. 1 in [9]):

**Theorem 1.1.** *Let  $M_k = \Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k)$  coming from a sequence of symmetric independent copies  $(\pi_j^k, \mathbb{C}, A_k, D_k)$  for  $-1 < q_k < 1$  and  $1 \leq k \leq n$ . Assume that for all  $1 \leq k \leq n$ ,  $H_k$  is finite dimensional and  $\dim_{\mathbb{C}}((D_k)_i(S_k)) < Cd^i$  for all  $i$  and some constants  $C, d > 0$ . Suppose that  $M = \overline{\bigotimes_{k=1}^n M_k} = N_1 \bar{\otimes} N_2$  for some type  $II_1$  factors  $N_1$  and  $N_2$ . Then there exists  $t > 0$  and a partition  $I_1 \sqcup I_2 = \{1, \dots, n\}$  such that, modulo conjugacy by an unitary in  $M$ , we have  $N_1^t = \overline{\bigotimes_{k \in I_1} M_k}$  and  $N_2^{1/t} = \overline{\bigotimes_{k \in I_2} M_k}$ .*

To prove Thm. 1.1, instead of relying on  $C^*$ -algebraic techniques and the property (AO) as in [8, 9], we use our relative strong solidity result in [2], Cor. 7.4. We should note that the von Neumann algebras  $\Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k)$  are automatically factors if  $\dim(H_k) \geq d(q_k)$  (Prop. 3.23 in [2]). By repeatedly applying Thm. 1.1 one obtains

**Corollary 1.2.** *Let  $M_k = \Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k)$  with the dimensions (over  $\mathbb{C}$ ) of the spaces  $(D_k)_i(S_k)$  sub-exponential and  $\infty > \dim(H_k) \geq d(q_k)$  for all  $1 \leq k \leq n$ . Assume that*

$$M_1 \bar{\otimes} \dots \bar{\otimes} M_n = N_1 \bar{\otimes} \dots \bar{\otimes} N_m,$$

*for  $m \geq n$  and some type  $II_1$  factors  $N_1, \dots, N_m$ . Then  $m = n$  and there exist  $t_1, \dots, t_n > 0$  with  $t_1 t_2 \dots t_n = 1$  such that, after permutation of indices and unitary conjugacy, we have  $N_k^{t_k} = M_k$ .*

When the factors  $N_j$  are assumed to be prime the assumption  $m \geq n$  becomes unnecessary and hence we obtain

**Corollary 1.3.** *Let  $M_1, \dots, M_n$  be generalized  $q$ -gaussians as above. Suppose that for some  $m \in \mathbb{N}$  and prime type  $II_1$  factors  $N_1, \dots, N_m$  we have*

$$M_1 \bar{\otimes} \dots \bar{\otimes} M_n = N_1 \bar{\otimes} \dots \bar{\otimes} N_m.$$

*Then  $m = n$  and there exist  $t_1, \dots, t_n > 0$  with  $t_1 t_2 \dots t_n = 1$  such that, after permutation of indices and unitary conjugacy, we have  $N_k^{t_k} = M_k$ . In particular this holds if each  $N_j = \Gamma_{q_j}(\mathbb{C}, T_j \otimes K_j)$  is a generalized  $q_j$ -gaussian with scalar coefficients, sub-exponential dimensions of  $(D_j)_i(T_j)$  and  $\dim(K_j) < \infty$ .*

In particular,  $M_k$  and / or  $N_j$  could be any one of the examples in 4.4.1, 4.4.2, 4.4.3 in [2]. Thus, if  $M_i, 1 \leq i \leq n$ , and  $N_j, 1 \leq j \leq m$ , are generalized  $q$ -gaussian von Neumann algebras as above and  $m \neq n$ , then  $M_1 \bar{\otimes} \dots \bar{\otimes} M_n \not\cong N_1 \bar{\otimes} \dots \bar{\otimes} N_m$ .

## 2. PROOF OF THE MAIN THEOREM.

Throughout this section, we freely use notations and results from Section 3 of [JungeUdreaGQC](#) by stating some preliminary technical results. The first one is Prop. 2.7 in [Povai](#). If  $(M, \tau)$  is a tracial von Neumann algebra and  $P, Q \subset M$  are von Neumann subalgebras, we say that  $P$  is amenable relative to  $Q$  (inside  $M$ ) if there exists a  $P$ -central state  $\Omega$  on  $B(L^2(M)) \cap (Q^{op})'$  such that  $\Omega|_M = \tau$  (see e.g. Def. 2.2 in [Povai](#)).

**Proposition 2.1.** *Let  $(M, \tau)$  be a tracial von Neumann algebras and let  $Q_1, Q_2 \subset M$  be von Neumann subalgebras. Assume that  $Q_1, Q_2$  form a commuting square, which means  $E_{Q_1} \circ E_{Q_2} = E_{Q_2} \circ E_{Q_1}$ , where  $E_{Q_1}, E_{Q_2}$  are the conditional expectations of  $M$  onto  $Q_1, Q_2$  respectively, and that  $Q_1$  is regular in  $M$ . Let  $P \subset M$  be a von Neumann subalgebra which is amenable relative to both  $Q_1$  and  $Q_2$ . Then  $P$  is amenable relative to  $Q_1 \cap Q_2$ .*

The next result is Prop. 12 in [OzawaPopaUPF](#).

**Proposition 2.2.** *Let  $M = M_1 \bar{\otimes} M_2$  and  $N \subset M$  be type  $II_1$  factors. Assume that  $N \prec_M M_1$  and  $N' \cap M$  is a factor. Then there exists a decomposition  $M = M_1^t \bar{\otimes} M_2^{1/t}$  for some  $t > 0$  and a unitary  $u \in \mathcal{U}(M)$  such that  $uNu^* \subset M_1^t$ .*

The next result will be needed in the proof of Thm. 1.1. It is an analogue of Prop. 15 in [OzawaPopaUPF](#). For convenience, if  $M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_n$  and  $1 \leq k \leq n$ , let's denote by

$$\widehat{M}_k = M_1 \bar{\otimes} \cdots \bar{\otimes} M_{k-1} \bar{\otimes} 1 \bar{\otimes} M_{k+1} \cdots \bar{\otimes} M_n \subset M.$$

More generally, for every subset  $I \subset \{1, \dots, n\}$ , we will denote by  $\widehat{M}_I$  the von Neumann algebra

$$\widehat{M}_I = \overline{\bigotimes_{i \notin I} M_i} \subset M.$$

**Proposition 2.3.** *Let  $M_i = \Gamma_{q_i}(\mathbb{C}, S_i \otimes H_i)$  be generalized  $q$ -gaussian von Neumann algebras with scalar coefficients coming from symmetric independent copies and having sub-exponential dimensions over  $\mathbb{C}$  of the spaces  $(D_i)_k(S_i)$ , for all  $1 \leq i \leq n$ . Let  $M = M_1 \bar{\otimes} \cdots \bar{\otimes} M_n$  and assume that  $N \subset M$  is a type  $II_1$  factor such that  $N' \cap M$  is a non-amenable factor. Then there exists  $t > 0$ ,  $1 \leq k \leq n$  and a unitary  $u \in \mathcal{U}(M)$  such that  $uNu^* \subset (\widehat{M}_k)^t$ .*

*Proof.* Let's first note that there exists a  $1 \leq k \leq n$  such that  $N' \cap M$  is not amenable relative to  $\widehat{M}_k$ . Indeed, if this were not the case, since the subalgebras  $\widehat{M}_I, \widehat{M}_J$  form a commuting square for all subsets  $I, J \subset \{1, \dots, n\}$  and all of them are regular in  $M$ , by repeatedly applying Prop. 2.1, we would obtain that  $N' \cap M$  is amenable relative to  $\bigcap_{k=1}^n \widehat{M}_k = \mathbb{C}$ , i.e.  $N' \cap M$  is amenable, a contradiction. Fix a  $k$  such that  $N' \cap M$  is not amenable relative to  $\widehat{M}_k$ . Suppose that  $N \not\prec_M \widehat{M}_k$ . By Cor. F.14 in [BrownOzawa](#) there exists an abelian von Neumann subalgebra  $\mathcal{A} \subset N$  such that  $\mathcal{A} \not\prec_M \widehat{M}_k$ . Let's make the following general remark. Suppose  $\Gamma_q(B, S \otimes H)$  is associated to a sequence of symmetric independent copies  $(\pi_j, B, A, D)$  and let  $\mathcal{M}$  be any tracial von Neumann algebra. Then

$$\mathcal{M} \bar{\otimes} \Gamma_q(B, S \otimes H) = \Gamma_q(B \bar{\otimes} \mathcal{M}, S \otimes H),$$

associated to a new sequence of symmetric independent copies  $(\tilde{\pi}_j, \tilde{B}, \tilde{A}, \tilde{D})$ , defined by  $\tilde{B} = B \bar{\otimes} \mathcal{M}$ ,  $\tilde{A} = A \bar{\otimes} \mathcal{M}$ ,  $\tilde{D} = D \bar{\otimes} \mathcal{M}$  and  $\tilde{\pi}_j : \tilde{A} \rightarrow \tilde{D}$  are given by  $\tilde{\pi}_j(a \otimes x) = \pi_j(a) \otimes x$ , for  $a \in A, x \in \mathcal{M}$ . Now note that

$$\mathcal{A} \subset M = \widehat{M}_k \bar{\otimes} M_k = \widehat{M}_k \bar{\otimes} \Gamma_{q_k}(\mathbb{C}, S_k \otimes H_k) = \Gamma_{q_k}(\widehat{M}_k, S_k \otimes H_k).$$

It's trivial to check that  $\dim_{\widehat{M}_k}((D_k)_i(S_k)) = \dim_{\mathbb{C}}(D_k)_i(S_k)$  are sub-exponential. Since  $\mathcal{A}$  is amenable relative to  $\widehat{M}_k$ , by applying Cor. 7.4 in [JungeUdreaGQC](#), we must have that either  $\mathcal{A} \prec_M \widehat{M}_k$  or  $N_M(\mathcal{A})''$  is amenable relative to  $\widehat{M}_k$ . The first half of the alternative is precluded by the choice

of  $\mathcal{A}$ , and the second would imply that  $N' \cap M \subset N_M(\mathcal{A})''$  is also amenable relative to  $\widehat{M}_k$ , which is a contradiction. Thus  $N \prec_M \widehat{M}_k$ , and by Prop. 2.2 we see that there exists a unitary  $u \in \mathcal{U}(M)$  and a  $t > 0$  such that  $uNu^* \subset (\widehat{M}_k)^t$ .  $\blacksquare$

Now we can prove Thm. 1.1. The proof proceeds verbatim as in [9]. We nevertheless give details for completeness.

*proof of Theorem 1.1.* We use induction over  $n$ . The case  $n = 0$  is trivial. Let  $M = \overline{\bigotimes_{k=1}^n M_k} = N_1 \bar{\otimes} N_2$ . Since  $M$  is non-amenable, we can assume that  $N_2$  is non-amenable. By Prop. 2.3 there exist  $t > 0$ ,  $1 \leq k \leq n$  and  $u \in \mathcal{U}(M)$  such that  $uN_1u^* \subset (\widehat{M}_k)^t$ . Set  $\mathcal{M}_1 = \widehat{M}_k$ ,  $\mathcal{M}_2 = M_k$  and  $N_{2,1} = N'_1 \cap u\mathcal{M}_1^t u^*$ . Then we see that

$$N_2 = N'_1 \cap M = u^*(N_{2,1} \bar{\otimes} \mathcal{M}_2^{1/t})u \subset u^*(\mathcal{M}_1^t \bar{\otimes} \mathcal{M}_2^{1/t})u = M,$$

and  $u\mathcal{M}_1^t u^*$  is generated by  $N_1$  and  $N_{2,1}$ . Using the induction hypothesis, we can find an  $s > 0$ , a partition  $I_1, I_{2,1}$  of  $\{1, \dots, n\} \setminus \{k\}$  such that  $N_1 = (\overline{\bigotimes_{j \in I_1} M_j})^s$  and  $N_{2,1} = (\overline{\bigotimes_{j \in I_{2,1}} M_j})^{t/s}$  after conjugating with a unitary element in  $(\widehat{M}_k)^t$ . If we now set  $I_2 = I_{2,1} \cup \{k\}$ , the proof is complete.  $\blacksquare$

**Remark 2.4.** *The statement of the results and the proofs remain verbatim the same if one assumes that  $B_k$  is a finite dimensional factor for every  $1 \leq k \leq n$ .*

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